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LETTER TO THE EDITOR

Polar decomposition of the twisted deRham complex for $\mathbb{C}_q$

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Abstract. A twisted exterior algebra for the complex quantum plane $\mathbb{C}_q$ in the polar coordinates is considered.

Recently a quantum deformation $\mathbb{C}_q$ of the complex plane $\mathbb{C}$ was introduced [1, 2] and intensively investigated [3, 4]. This quantum algebra is obtained from the Manin's plane [5] by means of the following definition of the star * antilinear anti-involution [1, 2]:

$$x^* = y \qquad y^* = x \tag{1}$$

where x and y are generators of the quantum plane. For completeness let as recall the definition of the Manin's plane [6]: a quantum plane is a quotient algebra $M_q^2 = \mathbb{C}(x, y)/\mathbb{I}(xy - qyx)$ where $\mathbb{C}(x, y)$ is an associative unital algebra over $\mathbb{C}$ freely generated by x and y, while $\mathbb{I}$ is the two-sided ideal in $\mathbb{C}$ spanned by monomials containing expressions

$$(xy - qyx)^k \qquad \text{with } k > 0 \text{ and } q \in \mathbb{C} - \{0\}.$$

In the following we will denote $x = \zeta$, $y = \zeta^*$, so in terms of ζ and ζ^*, the reordering rule has the form

$$\zeta\zeta^* = q\zeta^*\zeta. \tag{2}$$

Now, according to the general classification of the differential calculi related to *the quantum plane [2], the deRham complex associated with* $\mathbb{C}_q$ is parameter dependent and is generated by ζ, ζ^* and the differentials $\mathrm{d}\zeta$, $\mathrm{d}\zeta^*$ satisfying:

$$\begin{aligned} &\zeta\, \mathrm{d}\zeta = p\, \mathrm{d}\zeta\, \zeta && \zeta^*\, \mathrm{d}\zeta^* = p^{-1}\, \mathrm{d}\zeta^* \zeta^* \\ &\zeta^*\, \mathrm{d}\zeta = q^{-1}\, \mathrm{d}\zeta\, \zeta^* && \zeta\, \mathrm{d}\zeta^* = q\, \mathrm{d}\zeta^* \zeta \\ &\mathrm{d}\zeta\, \mathrm{d}\zeta^* = -q\, \mathrm{d}\zeta^*\, \mathrm{d}\zeta && (\mathrm{d}\zeta)^2 = (\mathrm{d}\zeta^*)^2 = 0. \end{aligned} \tag{3}$$

Here d is linear, nilpotent and satisfies the graded Leibnitz rule, while the parameter p belongs to $\mathbb{R}_+$. Moreover we can complete the rules (2) and (3) by derivatives ∂_ζ and ∂_{ζ^*} defined via

$$\mathrm{d}f(\zeta, \zeta^*) = \mathrm{d}\zeta\, \partial_\zeta f + \mathrm{d}\zeta^*\, \partial_{\zeta^*} f. \tag{4}$$

The reordering rules for ∂_ζ and ∂_{ζ^*} are listed in [2, 3].

Now, quantum complex analysis can be developed on the basis of the above twisted deRham algebra [3, 4]. In particular there exists a need to introduce an analogue of the polar coordinates into this algebra.

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To do this let us consider the following decomposition

$$\zeta = ru \tag{5}$$

where r is Hermitian while u is unitary, i.e.

$$r^* = r \qquad u^*u = uu^* = I. \tag{6}$$

The reordering rule for r and u reads

$$ur = q^{1/2} ru. \tag{7}$$

Notice that $r^2 = \zeta\zeta^*$ and equation (7) is in agreement with (2).

The derivation of the reordering rules for differentials (corresponding to equations (3)) is not so easy and demands some technical acrobatics. The final result is the following:

$$\begin{aligned} &u\,\mathrm{d}u = p^{1/2}\,\mathrm{d}u\,u \\ &u^*\,\mathrm{d}u = p^{-1/2}\,\mathrm{d}u\,u^* \\ &u\,\mathrm{d}r = (pq)^{1/2}\,\mathrm{d}r\,u \\ &u^*\,\mathrm{d}r = (pq)^{-1/2}\,\mathrm{d}r\,u^* \\ &r\,\mathrm{d}u = q^{-1/2}\,\mathrm{d}u\,r + (p^{1/2}-1)\,\mathrm{d}r\,u \\ &\mathrm{d}u\,r = (p^{-1/2}-1)u\,\mathrm{d}r + q^{1/2} r\,\mathrm{d}u \end{aligned} \tag{8a}$$

and

$$\begin{aligned} &r\,\mathrm{d}r = [(1+p^{3/2})/(p+p^{1/2})]\,\mathrm{d}r\,r + (1-p^{-1/2})\,\mathrm{d}u\,u^* r^2 \\ &\mathrm{d}r\,r = [(1+p^{3/2})/(p+p^{1/2})] r\,\mathrm{d}r + (1-p^{1/2}) r^2 u^*\,\mathrm{d}u. \end{aligned} \tag{8b}$$

Notice that equations ($8a$) have the *Bethe Ansatz form* [2] while the equations ($8b$) *are nonlinear in generators* with respect to the r. The remaining relations for the 2-forms read

$$\begin{aligned} &(\mathrm{d}u)^2 = 0 \\ &(\mathrm{d}r)^2 = (1-p^{-1/2})\,\mathrm{d}r\,\mathrm{d}u\,u^* r \\ &\mathrm{d}u\,\mathrm{d}r = -(pq)^{1/2}\,\mathrm{d}r\,\mathrm{d}u. \end{aligned} \tag{9}$$

Now, the polar representation of the $\mathbb{C}_q$ deRham complex is defined as a $\mathbb{C}_q$-module generated by r, u, $\mathrm{d}r$ and $\mathrm{d}u$.

The above differential calculus can be immediately completed by the derivatives ∂_r and ∂_u via the definition analogous to (4).

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